C. Chis, et al. Scientifical Researches. Agroalimentary Processes and Technologies, Volume XI, No. 2 (2005), 485-490

SOLVING NONLINEAR PROGRAMMING PROBLEMS BY LINEAR APPROXIMATIONS

Codruța Chiş[1], F. Creț[2]
[1]Faculty of Agriculture, Banat's University of Agricultural Sciences and Veterinary Medicine, Calea Aradului 119, RO-300645, Timişoara
[2]Faculty of Food Processing Technology, Banat's University of Agricultural Sciences and Veterinary Medicine, Calea Aradului 119, RO-300645, Timişoara

Abstract

By dividing the domain of a function into smaller intervals, one can find linear approximations of the function. It was used these to linearize a particular type of nonlinear programming problems, in order to use a modified simplex algorithm to give an approximate solution of such problems.

Keywords: *nonlinear programming, linear approximation*

Introduction

Suppose that $f:[a,b]\to IR$ (1) is a continuous function. For any $x\in[a,b]$, there is a unique $\lambda\in[0,1]$ such that

$$x=\lambda\cdot a+(1-\lambda)\cdot b \tag{2}$$

Denoting $\mu = 1 - \lambda$, one can then approximate the function f with

$$f(x)\approx\lambda\cdot f(a)+\mu\cdot f(b) \tag{3}$$

where $\lambda + \mu = 1$, which is linear with respect to λ and μ.

This approximation is efficient only for small intervals (Bellman, 1967; Marusciac, 1973). Generally, one can divide the interval [a, b] into a number of smaller intervals by choosing a division

$$a=x_0<x_1<\ldots<x_{k-1}<x_k<\ldots<x_{n-1}<x_n=b \tag{4}$$

Every point $x\in[a,b]$ can then be written as

$$x=\lambda_0\cdot x_0+\lambda_1\cdot x_1+\ldots+\lambda_n\cdot x_n \tag{5}$$

where $\lambda_i \in [0,1]$, for each $i = \overline{1,n}$, such that

$$\lambda_0 + \lambda_1 + \ldots + \lambda_n = 1 \tag{6}$$

and no more than two consecutive of the λ_i's are nonzero (to be precise, we have $\lambda_{k-1}, \lambda_k \neq 0 \Leftrightarrow x \in [x_{k-1}, x_k]$). The corresponding approximation of *f* is

$$f(x) \approx \lambda_0 \cdot f(x_0) + \lambda_1 \cdot f(x_1) + \ldots + \lambda_n \cdot f(x_n) \tag{7}$$

with

$$\lambda_0 + \lambda_1 + \ldots + \lambda_n = 1$$

and no more than two consecutive λ_i's nonzero (Malita, 1971).

Results and Discussion

Suppose now that

$$\begin{cases} \text{opt}[f(x_1, x_2, \ldots, x_m)] \\ g_1(x_1, x_2, \ldots, x_m) \leq (\geq) 0 \\ \vdots \\ g_r(x_1, x_2, \ldots, x_m) \leq (\geq) 0 \\ x_k \in [a_k, b_k], k = \overline{1,m} \end{cases} \tag{8}$$

is a programming problem with nonlinear objective function f and/or nonlinear restriction functions $g_1, \ldots, g_r$, such that

$$f(x_1, x_2, \ldots, x_m) = f_1(x_1) + f_2(x_2) + \ldots + f_m(x_m) \tag{9}$$

and

$$g_i(x_1, x_2, \ldots, x_m) = g_{i1}(x_1) + g_{i2}(x_2) + \ldots + g_{im}(x_m) \tag{10}$$

for any $i = \overline{1,r}$ (Postelnicu, 1977).

Linearizing each of the functions f_k, g_{ik} as described in the first section, could be obtain a linear programming problem:

$$\begin{cases} \quad \text{opt}[\lambda_{10} \cdot f_1(x_{10}) + \ldots + \lambda_{1n_1} \cdot f_1(x_{1n_1}) + \ldots \\ \quad \ldots + \lambda_{m0} \cdot f_m(x_{m0}) + \ldots + \lambda_{mn_m} \cdot f_m(x_{mn_m})] \\ \lambda_{10} \cdot g_{11}(x_{10}) + \ldots + \lambda_{1n_1} \cdot g_{11}(x_{1n_1}) + \ldots \\ \ldots + \lambda_{m0} \cdot g_{1m}(x_{m0}) + \ldots + \lambda_{mn_m} \cdot g_{1m}(x_{mn_m}) \le (\ge) 0 \\ \qquad \vdots \\ \lambda_{10} \cdot g_{r1}(x_{10}) + \ldots + \lambda_{1n_1} \cdot g_{r1}(x_{1n_1}) + \ldots \\ \ldots + \lambda_{m0} \cdot g_{rm}(x_{m0}) + \ldots + \lambda_{mn_m} \cdot g_{rm}(x_{mn_m}) \le (\ge) 0 \\ \lambda_{k0} + \lambda_{k1} + \ldots + \lambda_{kn_k} = 1, \lambda_{ki} \ge 0, k = \overline{1,m} \end{cases} \tag{11}$$

with the supplementary restriction that only two consecutive elements of each set of λ_{ki}'s, $k = \overline{1,m}$, may be nonzero.

One can solve this linear programming problem using a modified simplex algorithm, such that at each iteration step the supplementary restriction holds.

Of course, the optimal solution of the linear programming problem is not necessarily optimal for the initial nonlinear problem, but it represents, still, an approximation of an optimal solution. Other method may then allow to further optimizing this approximate solution.

In the sequel, we shall see, by hand of an example, how this linearization method works (Otiman, 2002; Vaduva, 1974).

Consider the following nonlinear programming problem

$$\begin{cases} \max[f(x_1, x_2) = 3x_1^3 + 2x_2^3] \\ \qquad x_1^2 + x_2^2 \le 16 \\ \qquad x_1 - x_2 \le 3 \\ \qquad x_1 \ge 0, x_2 \ge 0 \end{cases} \tag{12}$$

Under the given conditions, it is obvious that $x_1, x_2 \in [0,4]$. The problem could be linearised as described in the previous section, using the divisions of [0, 4]:

$$x_{10} = 0, x_{11} = 1, x_{12} = 2, x_{13} = 3, x_{14} = 4, \tag{13}$$

and

$$x_{20} = 0, x_{21} = 1, x_{22} = 2, x_{23} = 3, x_{24} = 4. \tag{14}$$

The objective function can be written as

$$f(x_1, x_2) = f_1(x_1) + f_2(x_2), \tag{15}$$

with: $f_1(x_1) = 3x_1^3$, $f_2(x_2) = 2x_2^3$.
Also

$$g_1(x_1, x_2) = x_1^2 + x_2^2 - 16 = g_{11}(x_1) + g_{12}(x_2) - 16, \tag{16}$$

where: $g_{11}(x_1) = x_1^2, g_{12}(x_2) = x_2^2$.

One obtains the linear programming problem:

$$\begin{cases} \max[z = 3\lambda_{11} + 24\lambda_{12} + 81\lambda_{13} + \\ +192\lambda_{14} + 2\lambda_{21} + 16\lambda_{22} + 54\lambda_{23} + 128\lambda_{24}] \\ \quad \lambda_{11} + 4\lambda_{12} + 9\lambda_{13} + 16\lambda_{14} + \\ \quad + \lambda_{21} + 4\lambda_{22} + 9\lambda_{23} + 16\lambda_{24} \le 16 \\ \quad \lambda_{11} + 2\lambda_{12} + 3\lambda_{13} + 4\lambda_{14} - \\ \quad - \lambda_{21} - 2\lambda_{22} - 3\lambda_{23} - 4\lambda_{24} \le 3 \\ \quad \lambda_{10} + \lambda_{11} + \lambda_{12} + \lambda_{13} + \lambda_{14} = 1, \\ \quad \lambda_{20} + \lambda_{21} + \lambda_{22} + \lambda_{23} + \lambda_{24} = 1, \\ \quad \lambda_{1j} \ge 0, \lambda_{2j} \ge 0, j = \overline{0,4}, \end{cases} \tag{17}$$

with the supplementary restriction, that no more than two consecutive of the λ_{1j} 's and λ_{2j} 's may be nonzero.

Using the modified simplex method, such that the supplementary condition is always fulfilled, one obtains the following solutions:

$$\lambda_{10} = 0, \lambda_{11} = 0, \lambda_{12} = 0, \lambda_{13} = \frac{1}{8}, \lambda_{14} = \frac{7}{8},$$

$$\lambda_{10} = \frac{1}{8}, \lambda_{11} = \frac{7}{8}, \lambda_{12} = 0, \lambda_{13} = 0, \lambda_{14} = 0,$$

for which

$$z_{max} = 81 \cdot \frac{1}{8} + 192 \cdot \frac{7}{8} + 0 \cdot \frac{1}{8} + 2 \cdot \frac{7}{8} = \frac{1439}{8}.$$

From this, was obtained an approximate solution of the initial nonlinear programming problem:

$$x_1 = 0 \cdot \lambda_{10} + 1 \cdot \lambda_{11} + 2 \cdot \lambda_{12} + 3 \cdot \lambda_{13} + 4 \cdot \lambda_{14} = \frac{31}{8},$$

$$x_2 = 0 \cdot \lambda_{20} + 1 \cdot \lambda_{21} + 2 \cdot \lambda_{22} + 3 \cdot \lambda_{23} + 4 \cdot \lambda_{24} = \frac{7}{8},$$

with the corresponding value of the objective function

$$f\left(\frac{31}{8}, \frac{7}{8}\right) = 3 \cdot \left(\frac{31}{8}\right)^3 + 2 \cdot \left(\frac{7}{8}\right)^3 = \frac{90059}{512}$$

Conclusions

By linearizing a nonlinear programming problem, one has the possibility of applying well-known algorithms, which can solve the associated linear programming problem. The optimal solution of the linear programming problem leads to a solution of the initial problem, which is not necessarily an optimal solution, but an approximation of the optimal one. The quality of the approximation depends on the approximation quality of the nonlinear functions:

$$f_1, \ldots, f_m, g_{11}, \ldots, g_{1m}, \ldots, g_{r1}, \ldots, g_{rm}.$$

This may increase if one refines the divisions of the intervals $[a_k, b_k], k = \overline{1, m}.$

The problem with such a refinement is that one obtains a large number of variables λ_{ki}. On the other hand, this is not much of a problem for good mathematical software, hence one can rely on computers to do the job and give a good approximate solution of the initial nonlinear programming problem.

References

Bellman, R.E., Dreyfus, S.E. (1967). *Programarea dinamică aplicată.* Editura Tehnică, Bucureşti

Malița, M., Zidăroiu, C. (1971). *Matematica organizării:* Editura Tehnică, Bucureşti

Maruşciac, I. (1973). *Metode de rezolvare a problemelor de programare neliniară.* Editura Dacia, Cluj

Otiman, I.P., Creț, F. (2002). *Elemente de Matematici aplicate în Economia agroalimentară.* Editura Agroprint, Timişoara

Postelnicu, T., Dinescu, C., Săvulescu, B. (1977). *Programări matematice şi Grafuri.* Editura didactică şi pedagogică, Bucureşti

Văduva, I., Dinescu, C., Săvulescu, B. (1974). *Modele matematice de organizarea şi conducerea producției.* Editura didactică şi pedagogică, Bucureşti