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APPROXIMATING FUNCTIONS WITH CHEBYSHEV POLYNOMIALS

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Abstract

In this paper, are reminded the most important, from the point of view of approximation theory, properties of Chebyshev polynomials and give a short Maple procedure to compute the coefficients of the Chebyshev-Fourier Series expansion of a continuous function on the interval [-1,1].

Key-words: *Chebyshev polynomials, Chebyshev-Fourier Series*

Introduction

One common way of approximating functions is to use Taylor series expansions. This relies on the computation of the Taylor polynomials of the function up to a certain order, and approximating the given function through these Taylor polynomials (Iaglom, 1983; Mocica, 1988). While this is a relatively simple procedure in case of smooth functions, it cannot work for nondifferentiable continuous functions. Also the convergence of these approximations is not uniformly distributed on a given interval, towards the ends of the intervals the approximation errors being higher (Panaitopol, 1980).

In order to avoid these problems, one can use different families of orthogonal polynomials – like Chebyshev's, Laguerre's, Legendre's or Hermite's (Dancea, 1973; Rudner, 1982). In the sequel, could be consider Chebyshev polynomials of the first kind (there is also a family of Chebyshev polynomials of the second kind).

Results and Discussion

The Chebyshev polynomials of the first kind are known also as the optimal approximation polynomials on the interval [-1, 1]. They are defined as

$$T_n(x) = \cos(n \cdot \arccos(x)), \qquad n \in IN, x \in [-1,1]. \tag{1}$$

With $x = \cos(\alpha)$, expanding $\cos(n \cdot \alpha)$, one can find that

$$T_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} C_n^{2k} \cdot x^{n-2k} \cdot \left(x^2 - 1\right)^k, \tag{2}$$

or

$$T_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k x^{n-2k} \cdot \sum_{l=k}^{\left[\frac{n}{2}\right]} C_n^{2k} \cdot C_l^k. \tag{3}$$

In order to compute the Chebyshev polynomials of the first kind one can use also Rodriguez's formula:

$$T_n(x) = (-1)^n \frac{\sqrt{1-x^2}}{(2n-1)!!} \cdot \frac{d^n}{dx^n}(1-x^2)^{n-\frac{1}{2}}, \tag{4}$$

or the generating function

$$\frac{1-tx}{1-2tx+t^2} = \sum_{n=0}^{\infty} T_n(x) \cdot t^n. \tag{5}$$

Another simple way of constructing Chebyshev's polynomials relies on the recurrence relation

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \qquad (\forall) n \geq 2, \tag{6}$$

starting with $T_0(x) = 1$, and $T_1(x) = x, (\forall) x \in [-1,1]$.

Most useful for the approximation theory is the fact that Chebyshev's polynomials of the first kind form a complete set of orthogonal polynomials with respect to the weight function $\rho(x) = \sqrt{1-x^2}$:

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \cdot T_m(x) \cdot T_n(x) dx = \begin{cases} \pi, m = n = 0 \\ \dfrac{\pi}{2}, m = n \neq 0 \\ 0, m \neq n \end{cases} \tag{7}$$

The Chebyshev polynomials expansion, or Chebyshev-Fourier series expansion, of a function f on the interval [-1, 1] is then given by

$$f(x) = \sum_{n=0}^{\infty} \alpha_n \cdot T_n(x), \tag{8}$$

where

$$\alpha_0 = \frac{1}{\pi} \cdot \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x) dx, \tag{9}$$

and

$$\alpha_n = \frac{2}{\pi} \cdot \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x) \cdot T_n(x) dx, \qquad (\forall) n > 0. \tag{10}$$

The following Maple procedure allows the computation of the Chebyshev-Fourier Series coefficients up to a certain specified order n, and computes the approximation of a function f using Chebyshev's polynomials up to order n.

```
>Cheb_approx:=proc(f,n) local F,i,a;
    F:=0;
    for i from 0 to n do
      if i=0
       then a:=1/Pi*int(1/sqrt(1-x^2)*f(x),x=-1..1)
       else a:=2/Pi*int(1/sqrt(1-x^2)*f(x)*orthopoly[T](i,x),x=-1..1)
      fi;
    F:=F+a*orthopoly[T](i,x)
    od;
    F
end;
```

Let us apply this procedure in order to compute the 5th Chebyshev approximation of the polynomial $X^7 + 2 \cdot X^4 + 3 \cdot X - 2$

```
> g:=unapply(Cheb_approx(x->x^7+2*x^4+3*x-2,5),x);
```

$$g := x \to -2 + \frac{199}{64}x - \frac{7}{8}x^3 + 2\,x^4 + \frac{7}{4}x^5$$

If is plotted (figure 1) the difference between the approximation function and the given polynomial:

```
plot(g-(x->x^7+2*x^4+3*x-2),-1..1);
```

could be seen that this difference is very small (in our case it's absolute value is less then 0.016)

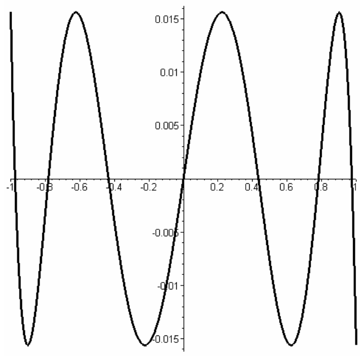

Fig. 1. Difference between the approximation function and given polynomial

Conclusions

The approximation of a function using Chebyshev series expansion is of real interest, known the fact that Chebyshev's polynomials are the polynomials of best approximation on the interval [-1, 1].

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